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TWO-FLUID MODEL OF WAVY SEPARATED TWO-PHASE FLOW

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Abstract—The objective of this study is to develop a local two-fluid model for the separated two-phase flow pattern, usually referred to as stratified flow. Previous models considered the stratified flow pattern as a superimposition of two single-phase flows. However, this assumption is valid for the cases in which the amplitude interfacial waves are small compared to the liquid thickness. In this paper, we propose a complementary approach for the case of thin films in comparison to the wavy region. In this case, a local two-fluid model accounting for the distribution of the two phases is necessary. The paper is based on one such local model of the separated two-phase flow pattern. Since the model does not predict the shape of the gas–liquid interface, we assume it is known *a priori*. The model accounts for the wavy surface and the interfacial transfer of momentum; this transfer can be induced both by pressure and viscous stress distributions along the wavy gas–liquid interface.

In the first part the mathematical development to establish the local two-fluid model of separated two-phase flow is presented. In the second part, the adequacy and advantages of simplifying the wave field by assuming a monochromatic dominant wave are considered. The closure conditions for the model are also presented. Interfacial terms of momentum transfer are shown to account for both the shape of the gas–liquid interface and for the distributions of stresses over it.

The key feature of the two-fluid model lies in the transfer of momentum at the wavy gas–liquid surface. The transfer of momentum at the gas–liquid interface raises two issues: the first is the deformation of the gas–liquid interface, the second is the distribution of the stresses over a wavy boundary (pressure and viscous stresses). The generation of waves, their deformation and propagation are beyond the scope of this work. In the second part of this paper, our goal is to adequately predict the effect of the distribution of the stresses over a wavy boundary for a given shape. In particular, the weight of the pressure term in the transfer of interfacial momentum is estimated. © 1997 Elsevier Science Ltd.

Key Words: two-fluid model, stratified flow, interfacial momentum transfer, liquid film

1. INTRODUCTION

Recent works on the two-fluid model in two-phase flows are mostly aimed at the dispersed flow pattern. However, separated two-phase flow behaviour is far from being understood. In particular, the momentum transfer between gas flow and a wavy liquid film is poorly predicted.

Two kinds of stratified flow model have been developed in the past: a global model and a complementary local approach. The objective of the global models is to calculate the pressure drop and the phase distribution in a stratified flow pattern in pipes. In one of the earlier developments of stratified flow models, Taitel and Dukler (1976) have proposed a model based on continuity and momentum equations for each phase, averaged over the pipe cross-section. Such a global approach can give useful information for applications, in particular for the design of pipelines for hydrocarbon transport in the oil industry. These models have been reviewed in recent papers (Jayanti 1991, Liné & Fabre 1996). It can be shown that the distribution of the phases results from momentum transfers at the pipe wall and at the interface between gas and liquid.

The closure problem is based on the modelling of momentum transfer coefficients at the wall and at the interface. Closure laws proposed by Andritsos and Hanratty (1987) result in a reliable

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global model. However, in the general review of Hanratty and McCready (1992), work on two-phase flow highlights the necessity of better understanding of small-scale interactions to improve the description of macroscopic behaviour. Hence, a complementary approach lies in the local modelling of two-phase stratified flow.

Several researchers have assumed that separated two-phase flow can be well represented by the superimposition of two single-phase flows separated by a flat interface. Akai *et al.* (1981) have proposed a local one-dimensional numerical model of stratified flow, to calculate the vertical profiles of longitudinal velocities in the gas and in the liquid, for steady-state and fully-developed flows. In their work, the authors solve a two-equation, low-Reynolds number model of turbulence, previously developed for single-phase flow by Jones and Launder (1992). This model is applied both to the gas and liquid simultaneously. However, the authors simplify the problem by considering an equivalent flat interface between gas and liquid: the equivalent interface accounts for the wavy two-phase flow region and the local interactions in this region. They impose specific boundary conditions on the interface. The applied boundary conditions provide a minimal physical relationship to physical information on the transfer of momentum at the gas–liquid boundary. In a similar approach, Issa (1988) has developed a 2-D numerical model of turbulent–turbulent stratified flow outside the wavy region by extrapolating the (k, ϵ) model of turbulence developed for single-phase flows. In a recent paper, Liné *et al.* (1996) presented a refined model for 3-D separated two-phase flow accounting for wave–mean flow interactions. The problem was solved outside the wavy region. The direction of turbulence modelling both above and below the waves remains unanswered.

The above mentioned approaches consider the superimposition of two single-phase flows. As mentioned earlier, this assumption is restricted to the case in which the wave region is small compared to the liquid thickness. In this paper, we propose a complementary approach in the case of thin films with relatively large wavy region. The present approach is based on local modelling of the separated two-phase flow pattern. Our formulation is based on the instantaneous local Eulerian equations of two-phase flows that are applied in the general frame of gas–liquid separated flows. The development is restricted to the particular case of stratified flow.

The primary objective of this paper is to establish the local two-fluid model and to analyse its ability to determine the unknowns of the problem: pressure, velocity fields and distribution of the phases. The second objective is to estimate the linear perturbation caused to the gas flow by the wavy interface. Since only linear perturbations are considered here, it seems sufficient to analyse the flow over a sinusoidal waveform. Given the deformation of the surface, we can express both the vertical profile of the phase fraction and the geometric parameters of the interfacial terms of momentum transfer. The balance equations for the two-fluid model are thus formulated. The contributions of the interfacial transfer of momentum along a given deformation interface appear explicitly in the momentum equations. A proposal to correlate the interfacial transfer due to the pressure variation with a form drag coefficient is presented. In conclusion the determination of phase distribution as governed by the momentum balance is also presented. Additional phenomenological relations are needed to deduce the geometrical characteristics of the waves from the volume fraction of the phase.

2. WAVY SEPARATED TWO-PHASE FLOW MODEL

The derivation of the two-fluid model is presented below. The local momentum balance is written in a general form. A simplified steady-state and fully develop flow model is then expressed, which will be discussed in a later section. The equations are finally cross-averaged to relate the global and local closure problems.

2.1. *Basic equations of motion in two-phase flow*

The objective of this paragraph is to define the variables commonly used for the local two-fluid model (Ishii 1975). In multiphase flows, the variables are continuous in the individual phases

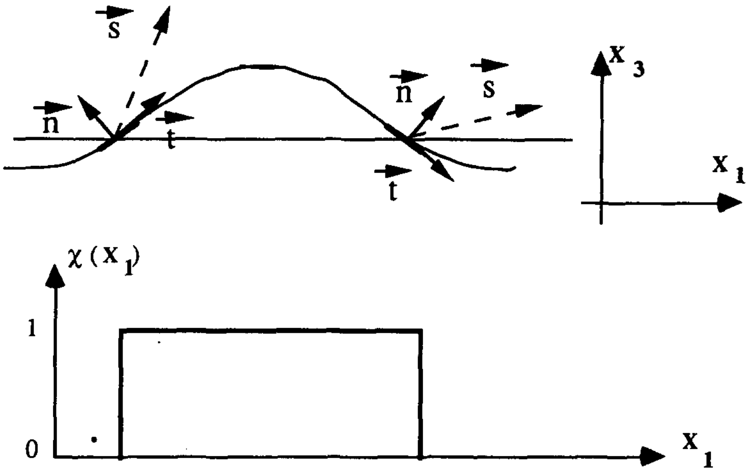

Figure 1. Definitions of stress, unit normal and unit tangential vectors at two geometrical locations having the same ordinate on the wavy interface and associated phase indicator function in the liquid.

but can be discontinuous at the interface. Therefore, a phase indicator function for each phase k must be introduced (figure 1):

$$\chi^k(x, t) = 1 \quad \text{if} \quad (x, t) \in \text{phase } k \qquad [1]$$
$$\chi^k(x, t) = 0 \quad \text{otherwise}$$

The phase indicator function for each phase k is continuous in the individual phases and discontinuous at the interface. The "material" derivative at the interface gives:

$$\frac{\partial \chi^k}{\partial t} + U_j^{\text{i}} \frac{\partial \chi^k}{\partial x_j} = 0 \qquad [2]$$

where U^{i} is the velocity of the interface. From geometrical consideration, one can write:

$$\frac{\partial \chi^k}{\partial x_j} = -n_j^k \delta^{\text{i}} \qquad [3]$$

where n^k is the unit vector normal to the interface and δ^{i} is the Dirac delta function.

The averaged variables are defined in each phase by:

$$\alpha^k \overline{\overline{V_j^k}} = \overline{\chi^k V_j^k} \quad \text{and} \quad \alpha^k \overline{\overline{P^k}} = \overline{\chi^k P^k} \qquad [4]$$

where V^k is the velocity, P^k the pressure, ρ^k the density and ν^k the kinemative viscosity of phase k respectively. Multiplying the Navier–Stokes equations by the phase indicator function and averaging leads to the classical form for the equations of motion in two-phase flow:

$$\frac{\partial \alpha^k}{\partial t} + \frac{\partial \alpha^k \overline{\overline{V_j^k}}}{\partial x_j} = -\overline{(V_j^k - U_j^{\text{i}}) n_j^k \delta^{\text{i}}} \qquad [5a \,\&\, b]$$

$$\frac{\partial \alpha^k \overline{\overline{V_j^k}}}{\partial t} + \frac{\partial \alpha^k \overline{\overline{V_m^k V_j^k}}}{\partial x_m} = -\frac{1}{\rho^k} \frac{\partial \alpha^k \overline{\overline{P^k}}}{\partial x_j} + \nu^k \frac{\partial}{\partial x_m}\left(\frac{\partial \alpha^k \overline{\overline{V_j^k}}}{\partial x_m} + \frac{\partial \alpha^k \overline{\overline{V_m^k}}}{\partial x_j}\right) + \frac{\partial \alpha^k \overline{\overline{R_{mj}^k}}}{\partial x_m} + \alpha^k g_j + \overline{L_j^k}$$

with the jump conditions:

$$\rho^{\text{G}} \overline{(V_j^{\text{G}} - U_j^{\text{i}}) n_j^{\text{G}} \delta^{\text{i}}} + \rho^{\text{L}} \overline{(V_j^{\text{L}} - U_j^{\text{i}}) n_j^{\text{L}} \delta^{\text{i}}} = 0 \quad \text{and} \quad \rho^{\text{G}} \overline{L_j^{\text{G}}} + \rho^{\text{L}} \overline{L_j^{\text{L}}} = 0 \qquad [5c \,\&\, d]$$

where the average of the phase indicator function is the volume fraction of phase k:

$$\alpha^k = \overline{\chi^k} \qquad [6]$$

In addition, one can derive:

$$\frac{\partial \alpha^k}{\partial x_j} = \overline{\frac{\partial \chi^k}{\partial x_j}} = -\overline{n_j^k \delta^{\mathrm{i}}} \tag{7}$$

The last term on the right-hand side of [5b], $\overline{L_j^k}$, represents the averaging of the interfacial momentum transfer. It requires further manipulations. The last term, $\overline{L_j^k}$, accounts for interfacial transfer of momentum and is written as follows:

$$L_j^k = -V_j^k(V_m^k - U_m^{\mathrm{i}})n_m^k\delta^{\mathrm{i}} - \frac{1}{\rho^k}\left[P^k\delta_{mj} - \mu^k\left(\frac{\partial V_j^k}{\partial x_m} + \frac{\partial V_m^k}{\partial x_j}\right)\right]n_m^k\delta^{\mathrm{i}} + v^k\left(\frac{\partial V_j^k n_m^k \delta^{\mathrm{i}}}{\partial x_m} + \frac{\partial V_m^k n_j^k \delta^{\mathrm{i}}}{\partial x_m}\right) \tag{8}$$

[5] and [8], previously derived by Lance *et al.* (1979), are the basis of any two-fluid model. In the past, the two-fluid model has only been applied to the dispersed two-phase flow pattern. In this study, it will be applied to the separated two-phase flow pattern.

In the regions above and below the waves, there are no interface and no gradient of phase fraction: single-phase flow can be observed. In the two-phase flow region, the phase fraction varies from one to zero and interfacial transfer occurs. The right-hand side of [5b] accounts for the interfacial transfer of mass and momentum: it has to be modelled. In a first analysis of the two-fluid model in separated phase wavy flow, we will focus on the closure of the viscous stress and pressure terms in the interfacial transfer.

The second term of [8] can be further expanded as follows:

$$\left[P^k\delta_{mj} - \mu^k\left(\frac{\partial V_j^k}{\partial x_m} + \frac{\partial V_m^k}{\partial x_j}\right)\right]n_m^k = P^k n_j^k - \tau_n^k n_j^k + \tau_t^k t_j^k \tag{9}$$

In the left-hand side of [9], the instantaneous stress vector is expressed with respect to a global frame of reference (figure 1). The right-hand side of the equation is expressed with respect to the normal and tangential unit vectors at the interface (figure 1). Some manipulations are needed to better understand the physical meaning of these terms.

In the case of a flat interface, the normal unit vector components are ($n_1 = 0$, $n_2 = 0$, $n_3 = 1$) and the tangential unit vector components are ($t_1 = 1$, $t_2 = 0$, $t_3 = 0$), Hence, in the longitudinal momentum balance equation (in the x_1 direction) there remains only the tangential viscous stress term. In the vertical momentum balance equation (in the x_3 direction) there remains only the normal viscous stress term and the pressure. In the case of a wavy interface, some manipulations are needed to better understand the physical meaning of these terms.

2.2. *Transformation of the pressure terms in the momentum balance*

Consider first the pressure terms in the right-hand side of [4]:

$$Pr_j^k = -\frac{1}{\rho^k}\frac{\partial \alpha^k \overline{\overline{P^k}}}{\partial x_j} - \frac{1}{\rho^k}\overline{P^k n_j^k \delta^{\mathrm{i}}} \tag{10}$$

Given P^{ref}, a reference value on the pressure, one can derive:

$$\frac{\partial \alpha^k}{\partial x_j}P^{\mathrm{ref}} = -\overline{n_j^k \delta^{\mathrm{i}} P^{\mathrm{ref}}} \tag{11}$$

which can be grouped and transformed as:

$$Pr_j^k = \underset{\text{(i)}}{-\frac{\alpha^k}{\rho^k}\frac{\partial \overline{\overline{P^k}}}{\partial x_j}} \underset{\text{(ii)}}{- \frac{1}{\rho^k}\frac{\partial \alpha^k}{\partial x_j}(\overline{\overline{P^k}} - P^{\mathrm{ref}})} \underset{\text{(iii)}}{- \frac{1}{\rho^k}\overline{(P^k - P^{\mathrm{ref}})n_j^k\delta^{\mathrm{i}}}} \tag{12}$$

where: (i) is a classical pressure gradient in phase k; (ii) accounts for the difference between the local phasic pressure and a reference pressure; (iii) accounts for the difference between the local interfacial pressure and the reference pressure.

If we simplify the problem by considering a one-dimensional problem of steady-state fully-developed (in the x_1 direction) separated two-phase flow, the longitudinal momentum equation enables us to calculate the vertical profiles of pressure and longitudinal velocity. At a given vertical ordinate x_3, the preceding term can be written as:

$$Pr_1^k = -\frac{\alpha^k}{\rho^k}\frac{\partial \overline{\overline{P^k}}}{\partial x_1} - \frac{1}{\rho^k}\overline{(P^k - P^{\mathrm{ref}})n_1^k\delta^{\mathrm{i}}} \qquad [13]$$

where $\overline{\overline{P^k}}$ is the local value of the pressure after averaging over a set of measurements taken in the phase k at the ordinate x_3, and P^k is the local instantaneous value of the pressure taken each time that the gas–liquid interface crosses the ordinate x_3 during the ensemble averaging operation. At any given time, the interfacial pressure has a different value and the shape of the interface is different: i.e. during each measurement the unit normal vector perpendicular to the interface has a particular orientation. Term (iii) accounts for both the modulus of pressure and the orientation of the unit vector.

An initial attempt to define the reference pressure would be to choose the overall average of the modulus of pressure taken each time that the gas–liquid interface crosses the ordinate x_3:

$$P^{\mathrm{ref}} = \overline{P^k\delta^{\mathrm{i}}} \qquad [14a]$$

In the proposed, we will consider the separated flow pattern and assume a sinusoidal perturbation of the interface. It seems useful to choose as reference pressure the mean value of the pressure along the interface:

$$P^{\mathrm{ref}} = \langle P^{\mathrm{i}} \rangle \qquad [14b]$$

The pressure term can be written as:

$$Pr_j^k = -\frac{\alpha^k}{\rho^k}\frac{\partial \overline{\overline{P^k}}}{\partial x_j} - \frac{1}{\rho^k}\frac{\partial \alpha^k}{\partial x_j}(\overline{\overline{P^k}} - \langle P^{\mathrm{i}}\rangle) - \frac{1}{\rho^k}\overline{\tilde{P}n_j^k\delta^{\mathrm{i}}} \qquad [15]$$

Each term of the momentum balance equation can be expressed in the same way.

2.3. *Expression of momentum balance in separated two-phase flow*

After a series of mathematical manipulations, the momentum balance in separated two-phase flow can be written as follows:

$$\alpha^k\left[\frac{\partial \overline{\overline{V_j^k}}}{\partial t} + \frac{\partial \overline{\overline{V_m^k}}\,\overline{\overline{V_j^k}}}{\partial x_m} + \frac{1}{\rho^k}\frac{\partial \overline{\overline{P^k}}}{\partial x_j} - v^k\frac{\partial}{\partial x_m}\left(\frac{\partial \overline{\overline{V_j^k}}}{\partial x_m} + \frac{\partial \overline{\overline{V_m^k}}}{\partial x_j}\right) - \frac{\partial \overline{\overline{R_{mj}^k}}}{\partial x_m} - g_j\right]$$

$$+ \overline{\overline{V_j^k}}\frac{\partial \alpha^k}{\partial t} + (\overline{\overline{V_m^k}}\,\overline{\overline{V_j^k}} - \overline{\overline{R_{mj}^k}})\frac{\partial \alpha^k}{\partial x_m} + \frac{1}{\rho^k}(\overline{\overline{P^k}} - \langle P^{\mathrm{i}} - \tau_n^{\mathrm{i}}\rangle)\frac{\partial \alpha^k}{\partial x_j} - \frac{\partial \alpha^k}{\partial x_m}v^k\left(\frac{\partial \overline{\overline{V_j^k}}}{\partial x_m} + \frac{\partial \overline{\overline{V_m^k}}}{\partial x_j}\right) - \frac{1}{\rho^k}\langle \tau_t^{\mathrm{i}}\rangle\overline{t_j^k\delta^{\mathrm{i}}}$$

$$- v^k\left[(\overline{\overline{V_j^k}} - \langle V_j^{\mathrm{i}}\rangle)\frac{\partial^2\alpha^k}{\partial x_m^2} + (\overline{\overline{V_m^k}} - \langle V_m^{\mathrm{i}}\rangle)\frac{\partial^2\alpha^k}{\partial x_m\partial x_j}\right] - v^k\left[\frac{\partial \alpha^k}{\partial x_m}\frac{\partial(\overline{\overline{V_j^k}} - \langle V_j^{\mathrm{i}}\rangle)}{\partial x_m} + \frac{\partial \alpha^k}{\partial x_j}\frac{\partial(\overline{\overline{V_m^k}} - \langle V_m^{\mathrm{i}}\rangle)}{\partial x_m}\right]$$

$$= -\overline{V_j^k(V_m^k - U_m^{\mathrm{i}})n_m^k\delta^{\mathrm{i}}} - \frac{1}{\rho^k}[\overline{\tilde{P}n_j^k\delta^{\mathrm{i}}} - \overline{\tilde{\tau}_n n_j^k\delta^{\mathrm{i}}} - \overline{\tilde{\tau}_t t_j^k\delta^{\mathrm{i}}}] + v^k\left(\frac{\partial \overline{V_j^k n_m^k\delta^{\mathrm{i}}}}{\partial x_m} + \frac{\partial \overline{V_m^k n_j^k\delta^{\mathrm{i}}}}{\partial x_m}\right) \qquad [16]$$

In this equation: (i) for the volume fraction α^k of phase k times the equivalent single phase flow equation; in the regions above and below the waves, the value of the volume fraction α^k is one and thus reduces the equation to its single-phase form; (ii) the second part of the left-hand side contains both averaged variables and derivatives of the volume fraction of phase k. Given the shape of the gas–liquid interface, each term can be evaluated. The mean values of both pressure and viscous stress along the interface ($\langle P^{\mathrm{i}} - \tau_{\mathrm{n}}^{\mathrm{i}}\rangle\langle\tau_{\mathrm{t}}^{\mathrm{i}}\rangle$) do appear explicitly; (iii) the right-hand side of the equation accounts for the interfacial transfer of momentum due to the mass transfer as well as the pressure and viscous stress perturbations along the interface $[\overline{\tilde{P}n_j^k\delta^{\mathrm{i}}} - \overline{\tilde{\tau}_{\mathrm{n}}n_j^k\delta^{\mathrm{i}}} - \overline{\tilde{\tau}_{\mathrm{t}}t_j^k\delta^{\mathrm{i}}}]$.

The turbulent terms $\overline{\overline{R_{ij}^k}}$ in [16] require additional closure terms to adequately model the turbulence phenomena. In this paper, the turbulence problem is not studied. Attention is restricted to the interfacial transfer terms in wavy separated two-phase flows.

2.4. *1-D local model of separated two-phase flow*

In order to simplify the analysis, we can consider a one-dimensional problem corresponding to steady-state fully-developed (in the x_1 direction) separated two-phase flow without mass transfer. The unknowns of the problem are then the vertical profiles of longitudinal velocity $\overline{\overline{V_1^k}}(x_3)$, pressure $\overline{\overline{P^k}}(x_3)$, the distribution of the phases $\alpha^k(x_3)$ and the longitudinal pressure gradient. The closure problem is once more related to the interfacial terms of momentum transfer which are required to express the shape of the interface and the distributions of interfacial stresses.

The longitudinal momentum equation can be written as:

$$\alpha^k\left[\frac{1}{\rho^k}\frac{\partial\overline{\overline{P^k}}}{\partial x_1}-\nu^k\frac{\partial^2\overline{\overline{V_1^k}}}{\partial x_3}-\frac{\partial\overline{\overline{R_{31}^k}}}{\partial x_3}-g_1\right]-\frac{\partial\alpha^k}{\partial x_3}\left[\nu^k\frac{\partial\overline{\overline{V_1^k}}}{\partial x_3}+\overline{\overline{R_{31}^k}}\right]-\frac{1}{\rho^k}\langle\tau_1^{\mathrm{i}}\rangle\overline{t_1^k\delta^{\mathrm{i}}}-\nu^k(\overline{\overline{V_1^k}}-\langle V_1^{\mathrm{i}}\rangle)\frac{\partial^2\alpha^k}{\partial x_3^2}$$

$$-\nu^k\frac{\partial\alpha^k}{\partial x_3}\frac{\partial(\overline{\overline{V_1^k}}-\langle V_1^{\mathrm{i}}\rangle)}{\partial x_3}=-\frac{1}{\rho^k}[\overline{\tilde{P}n_1^k\delta^{\mathrm{i}}}-\overline{\tilde{\tau}_n n_1^k\delta^{\mathrm{i}}}-\overline{\tilde{\tau}_t t_1^k\delta^{\mathrm{i}}}]+\nu^k\left(\frac{\partial\overline{V_1^k n_3^k\delta^{\mathrm{i}}}}{\partial x_3}+\frac{\partial\overline{V_3^k n_1^k\delta^{\mathrm{i}}}}{\partial x_3}\right)\quad[17]$$

The vertical momentum equation writes then as:

$$\alpha^k\left[\frac{1}{\rho^k}\frac{\partial\overline{\overline{P^k}}}{\partial x_3}-\frac{\partial\overline{\overline{R_{33}^k}}}{\partial x_3}-g_3\right]-\frac{\partial\alpha^k}{\partial x_3}+\overline{\overline{R_{33}^k}}\frac{1}{\rho^k}(\overline{\overline{P^k}}-\langle P^{\mathrm{i}}\rangle+\langle\tau_n^{\mathrm{i}}\rangle)\frac{\partial\alpha^k}{\partial x_3}-\frac{1}{\rho^k}\langle\tau_t^{\mathrm{i}}\rangle\overline{t_3^k\delta^{\mathrm{i}}}$$

$$+2\nu^k\langle V_3^{\mathrm{i}}\rangle\frac{\partial^2\alpha^k}{\partial x_3^2}+2\nu^k\frac{\partial\alpha^k}{\partial x_3}\frac{\partial\langle V_3^{\mathrm{i}}\rangle)}{\partial x_3}=-\frac{1}{\rho^k}[\overline{\tilde{P}n_3^{\mathrm{i}}\delta^{\mathrm{i}}}-\overline{\tilde{\tau}_n n_3^{\mathrm{i}}\delta^{\mathrm{i}}}-\overline{\tilde{\tau}_t t_3^{\mathrm{i}}\delta^{\mathrm{i}}}]+2\nu^k\frac{\partial\overline{V_3^k n_3^{\mathrm{i}}\delta^{\mathrm{i}}}}{\partial x_3}\quad[18]$$

2.5. *Cross-averaged model of separated two-phase flow*

In many industrial applications, cross-averaged models are used. Cross-averaging the equations and unknowns over the section of the pipe, we define the new variables:

$$\epsilon^k=\frac{1}{A}\int_A\alpha^k\,\mathrm{d}a\quad\text{and}\quad\epsilon^k\overline{V_1^{kA}}=\frac{1}{A}\int_A\alpha^k\overline{\overline{V_1^k}}\,\mathrm{d}a\quad[19]$$

We consider a steady-state fully-developed (in x_1) separated two-phase flow without mass transfer. After simplifications relative to fully-developed and steady-state stratified flow, the longitudinal balance equation of momentum can be written as:

$$\epsilon\frac{\partial\overline{P^{kA}}}{\partial x_1}=\epsilon^k\rho^k g_1+\frac{S^{wk}\overline{\tau^{kw}}}{A}+\frac{S^{\mathrm{i}}\overline{\tau^{ki}}}{A}-\frac{S^{\mathrm{i}}\overline{\pi^{ki}}}{A}\quad[20]$$

where $\overline{\tau^{kw}}$ is the mean shear stress at the wall wetted by the phase k on the perimeter S^{wk}; $\overline{\tau^{ki}}$ is the mean shear stress at the interface on the perimeter S^{i}; and $\overline{\pi^{ki}}$ is the normal stress contribution (pressure − normal viscous stress) at the interface on the perimeter S^{i}.

Usually, only the first term of interfacial momentum transfer is considered. This term is exact only if the gas–liquid interface is flat. The longitudinal component of the interfacial stress vector corresponds to the interfacial shear stress. But, with the presence of the waves, the averaged value of the interfacial distribution of stress vector can have a longitudinal component which accounts both for viscous stress and pressure contributions.

3. PARTICULAR CASE OF MONOCHROMATIC WAVY INTERFACE

In the following text, we consider a sinusoidal shaped interface. This is convenient both for imposing the profile of phase fraction and for developing linear analysis. In the case of a small amplitude 2-D wave, the interfacial stress perturbations are theoretically and experimentally given by their amplitude and phase shift.

The analysis that is proposed in this paper is restricted to a separated two-phase flow pattern. The shape of the gas–liquid interface will be purely sinusoidal. It is based on the assumption of a slow varying dominant 2-D wave. In the 2-D wave stratified flow pattern, the wave energy spectrum shows the appearance of a characteristic dominant wave. The turbulent gas flow over the waves generates ripples; their wavelengths are smaller in comparison to the dominant wave. Such a wave field will be considered as the superposition of a dominant wave and smaller scale waves or ripples. The dominant wave will be represented by a sinusoidal monochromatic gravity wave (figure 1).

Indeed, the dominant wave can be considered as slowly varying. The variations of the dominant wave are governed by non-linear energy transfer between wave components. van Gastel (1987) has shown that the time-scale t linked to these non-linear interactions can be estimated as:

$$t = \frac{T}{(ak)^2} \tag{21}$$

where T stands for the period of the dominant wave. Given a dominant wave frequency of 10 Hz and a wave steepness of 10%, the previous relation gives $t = 25$ s. It is quite a large time scale; hence, the dominant wave can be thought of as slowly varying. This remark validates the idea of steady-state fully-developed stratified flow.

In addition, Belcher and Hunt (1993) have estimated the growth time scale $t(k)$ of a wave of wave number k as:

$$t(k) = \frac{\rho_L}{\rho_G} \frac{c^2}{V^{*2}} \frac{1}{ck} \tag{22}$$

Ripples have very large wave numbers; hence they grow on much smaller time scales than the dominant wave. Following Belcher and Hunt, we consider that the ripple field is in local equilibrium with the turbulent gas flow. These ripples will behave as roughness to the gas flow. One can refer to the Charnock (1995) formula to estimate the scale of the ripples for a sheared gas flow corresponding to the friction velocity V^*.

Consequently, in this paper the 2-D wave field is composed of a given dominant wave over which ripples can be superimposed. The local and instantaneous position of the dominant shape of the interface, relative to the mean liquid thickness, is given by:

$$\zeta = \mathrm{Re}(\hat{\zeta} e^{i\varphi}) = a \cos \varphi \tag{23}$$

where $\hat{\zeta}$ or a is the amplitude of the sinusoidal perturbation (real number) and φ its phase.

Given this shape, one can derive the vertical profile of the liquid fraction:

$$\alpha_L(x_3) = \frac{1}{\pi} \operatorname{acos}\left(\frac{x_3 - h_L}{a}\right) \quad \text{if} \quad -a < x_3 - h_L < a \tag{24}$$

$$\alpha_L(x_3) = 1 \quad \text{if} \quad -a \geqslant x_3 - h_L \quad \text{and} \quad \alpha_L(x_3) = 0 \quad \text{if} \quad x_3 - h_L \geqslant a$$

The interest of such a shape is its simplicity. It will enable us to develop linear analysis. However, this shape corresponds to a discontinuous slope when $x_3 - h_L = \pm a$. In addition, it must be emphasized that, given the mean level and the shape of the wavy interface, it is possible to derive the volume fraction of each phase (see [24]). As we will see in the next part of this paper, this distribution must be compatible with the momentum balance. It seems important to indicate that, even if the two-fluid model enables us to calculate the profile of the volume fraction of each phase, it is impossible to deduce the exact form of the wavy interface from this profile without any additional assumption on the phase distributions.

Given this shape, any vertical position x_3 will intercept the interface at well-defined instants. It corresponds to pairs of events as shown in figure 1. For small wave amplitude, slope and celerity, flow perturbations are calculated as linear responses to the sinusoidal boundary perturbation. In the frame of linear analysis, the complex perturbations of pressure and viscous stresses will be taken both sinusoidally and phase shifted. The total stress can be expressed in pressure and viscous stress components. In addition, the viscous stress component is written in terms of tangential and normal component along the interface. These expressions can be written as follows:

$$P^{i} = \langle P^{i} \rangle + \mathrm{Re}(\tilde{P}) \quad \text{with} \quad \tilde{P} = |\hat{P}|e^{i(\varphi + \theta_P)} \qquad [25a]$$

$$\tau_t^i = \langle \tau_t^i \rangle + \mathrm{Re}(\tilde{\tau}_t) \quad \text{with} \quad \tilde{\tau}_t = |\hat{\tau}_t|e^{i(\varphi + \theta_t)} \qquad [25b]$$

$$\tau_n^i = \langle \tau_n^i \rangle + \mathrm{Re}(\tilde{\tau}_n) \quad \text{with} \quad \tilde{\tau}_n = |\hat{\tau}_n|e^{i(\varphi + \theta_n)} \qquad [25c]$$

where the first bracketed terms are averaged along the interface and the second bracketed values with tilda terms are the complementary perturbations induced by the wavy surface, $|\hat{P}|$, $|\hat{\tau}_t|$ and $|\hat{\tau}_n|$ are the moduli of these pressure, tangential and normal viscous stress perturbations and θ_P, θ_t and θ_n their phase shifts respectively.

If the normal component of the stress perturbation is either in phase or out of phase with the sinusoidal boundary, each pair of events will have a null contribution to the longitudinal balance equation. On the contrary, any phase shift between the normal stresses and the sinusoidal boundary will lead to a non-zero term of interfacial transfer of momentum.

Figure 2 shows the contribution of the wavelength averaged interfacial pressure $\langle P^i \rangle$ to the momentum balance equation. The two longitudinal components are equal in amplitude and opposite in sign; their horizontal resultant is null; it does not contribute to the longitudinal momentum balance. The two vertical components are equal in amplitude and have the same sign; their vertical resultant is not null; it contributes to the vertical momentum balance.

Figure 3 shows the contribution of the wavelength averaged interfacial tangential shear stress $\langle \tau_t^i \rangle$ to the momentum balance. The two longitudinal components are equal in amplitude and have the same sign; their horizontal resultant is not null; it contributes to the longitudinal momentum balance. The two vertical components are equal in amplitude and opposite in sign; their vertical resultant is null; it does not contribute to the vertical momentum balance.

Figure 4 shows the contribution of the perturbed interfacial pressure variation $\tilde{P}$ to the momentum balance. The two longitudinal components are generally not equal in amplitude but can have the same sign; their horizontal resultant is generally not null; it can contribute to the longitudinal momentum balance. The two vertical components are generally not equal in amplitude but can have the same sign; their vertical resultant is generally not null; it can contribute to the vertical momentum balance.

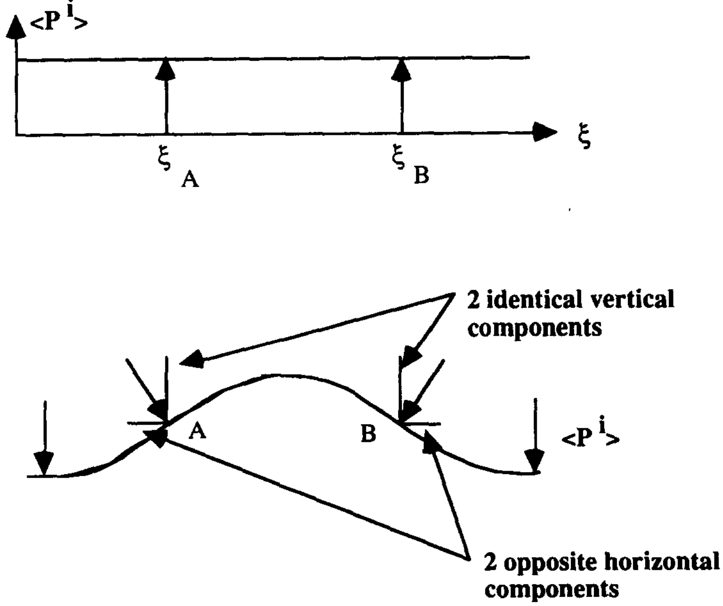

Figure 2. Contribution of the wavelength averaged interfacial pressure to the horizontal and vertical momentum balance.

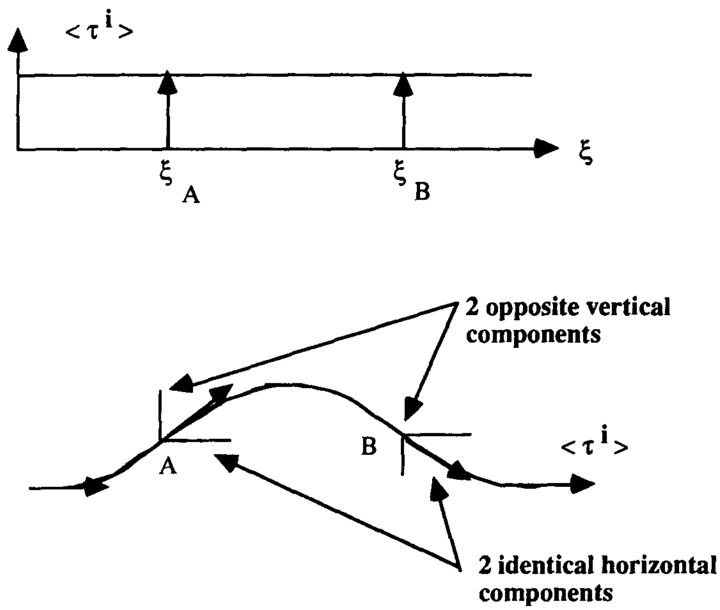

Figure 3. Contribution of the wavelength averaged interfacial shear stress to the horizontal and vertical momentum balance.

In the literature, a large amount of data has been presented for the case of single phase flow over a sinusoidally shaped solid boundary. Consider first these results.

Zilker and Hanratty (1979) review the data over stationary sinusoidal waves. The analysis of the results shows that the modulus of the tangential viscous stress perturbation (Abrams 1984) is at least one order of magnitude smaller than the pressure perturbation one. The cosine and sine functions being bounded by ± 1, the tangential term will play a minor role compared to the pressure term in the longitudinal momentum balance. From a physical point of view, it is clear that basically the viscous effects are responsible for the phase shift of the pressure perturbation; in the case of inviscid fluid, one can derive analytically that the pressure perturbation is exactly out of phase with the boundary perturbation. A phase shift of 180° was obtained by Helmholtz (1868) for inviscid fluid flow.

Benjamin (1959) accounted for the non-uniform velocity profile and viscosity. The study considered a viscous fluid flowing at a velocity V_∞, above a sinusoidally shaped wall (amplitude a and wavenumber k). The derived expressions can be written as shown:

$$|\hat{P}|\cos(\theta_P) = b(kx, \mathrm{Re}_x)ak\rho V_\infty^2 \qquad [26a]$$

$$|\hat{P}|\sin(\theta_P) = s(kx, \mathrm{Re}_x)ak\rho V_\infty^2 \qquad [26b]$$

where s is a sheltering coefficient introduced a long time ago by Jeffreys (1925). Benjamin (1959) showed that the sheltering coefficient depends on kx, where x is the abscissa in the developing boundary layer and on the Reynolds number $\mathrm{Re} = xV_\infty/\nu$. One can easily deduce from these expressions the amplitude and phase shift of the pressure perturbation.

In order to better account for the effect of turbulence, several numerical codes have been developed. In the frame of the linear analysis, it is possible to express the equations of the perturbed motion in terms of a modified Orr–Sommerfeld equation (Abrams 1984, Lopez 1994). The numerical solution of the problem gives the perturbed velocity and pressure fields. One can then derive the pressure and viscous stress perturbations at the boundaries. The weak point of these models remains however the modelling of the turbulence. Belcher *et al.* (1993) proposed a criteria on that the turbulence model must satisfy to correctly predict the perturbed flow with rapidly varying pressure gradient. The approach of Belcher (1990) based on the rapid distortion theory of turbulence should be applied to internal separated

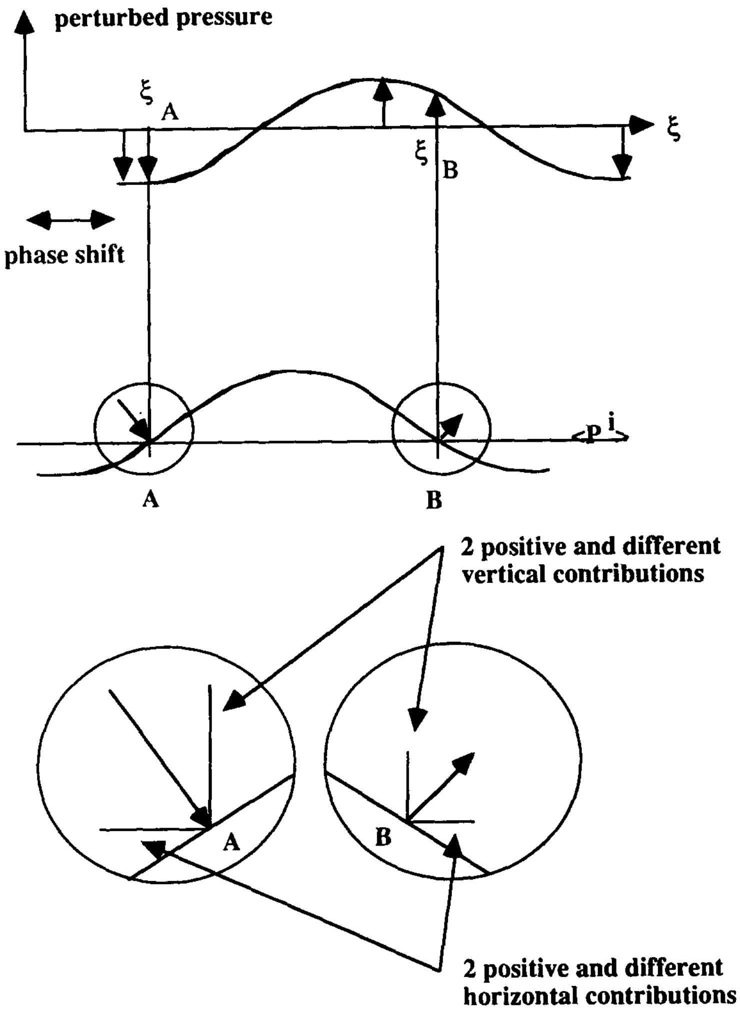

Figure 4. Contribution of the perturbed interfacial pressure variation to the horizontal and vertical momentum balance.

two-phase flows. In this view, the most promising way of research is probably related to direct numerical simulation and confirmation of results with particle image velocimetry measurements.

Experimental data of amplitude and phase shift of pressure perturbation are also available in the case of mechanically generated waves sheared by a gas flow (Shemdin & Hsu 1965, Papadimitrakis *et al.* 1986).

4. CLOSURE PROBLEM OF INTERFACIAL TRANSFER IN WAVY SEPARATED TWO-PHASE FLOWS

In this part, we derive the exact form of the perturbed pressure contribution to the interfacial transfer of momentum. We propose to express this interfacial transfer in terms of a form drag coefficient. This coefficient is correlated using experimental data for the amplitude and phase shift of the pressure variations above solid wavy walls and a wavy gas–liquid interface. We then return to the local two-fluid model of separated two-phase flow. Assuming fully-developed and steady-state flow conditions, we show how to determine the phase distribution from the momentum balance. However, it is important to note that it does not give the exact form of the interface.

4.1. *Interfacial momentum transfer due to the pressure fluctuations*

Given the preceding definitions, the interfacial momentum transfer terms of the longitudinal momentum balance equation can be analytically expressed. Considering the dominant role of the term related to the pressure perturbations, we will focus on it. After certain mathematical manipulations, it can be shown that:

$$\overline{Pn_1^i\delta^i}(x_3) = -\frac{2}{\lambda}|\hat{P}|\sin(\theta_P)\sqrt{1-\left(\frac{x_3-h_L}{a}\right)^2} \quad \text{if} \quad -a < x_3 - h_L < a \qquad [27]$$

It is clear that in the frame of our assumptions, the pressure term vanishes when it is in phase or out of phase with the boundary perturbation. The normal viscous stress behaves identically. However, the tangential viscous stress term vanishes when it is quadratic with the boundary perturbation. It is important to highlight that only the imaginary part of the pressure perturbation appears in the expression of the interfacial momentum transfer.

It is interesting to average the momentum equation over the section of the flow. In particular, the cross-averaged interfacial pressure term $(\overline{\pi^{ki}}$ writes as:

$$\overline{\pi^{ki}} = -\frac{ak}{2}|\hat{P}|\sin\theta_P \qquad [28]$$

In addition, the cross-averaged interfacial term for the tangential viscous stress perturbation can be shown to be exactly equal to zero. Only the mean value of the shear stress averaged over the interface $\overline{\tau^{ki}}$ remains. Once more, the imaginary part of the pressure perturbation appears in the interfacial momentum transfer term.

We introduce a drag form coefficient defined as:

$$C_D = \frac{\frac{ak}{2}|\hat{P}|\sin\theta_P}{\frac{1}{2}\rho V_\infty^2} = ak\frac{1}{\rho}|\hat{P}|\sin\theta_P \qquad [29]$$

which is consistent with the form proposed by Benjamin and Jeffreys presented below:

$$C_D = s(ak)^2 \qquad [30]$$

Consequently, the closure problem on the interfacial momentum transfer can be equivalently expressed in terms of the imaginary part of the pressure (which needs to derive both the amplitude and the phase shift of the pressure perturbation) or in terms of a form drag coefficient.

4.2. *Strategy to close the pressure term*

The amplitude and the phase shift can be derived by solving the perturbed flow problem. Up to now, there remain many unknowns in the nature of the turbulence above and below the waves. Therefore, oversimplified turbulent modelling would lead to unphysical results.

Consider the trend shown by the experimental values of the amplitudes and phase shifts of the stress perturbations. Given the previously referred to experiments over a smooth solid wall sinusoidally shaped (amplitude a and wavenumber k), one can estimate from experiments the form drag coefficient:

$$C_D = ak\frac{1}{\rho}|\hat{P}|\sin\theta_P \qquad [31]$$

The analysis must be restricted to small wave steepness, in order to obtain a linear response in the stress variations with a single harmonic. In addition, the gas flow must not separate from the downstream side of the wavy wall. Considering that the wall is smooth and that the wave amplitude is small, one can estimate the friction factor f_{smooth} by extrapolating the classical Blasius formula. In order to evaluate the respective weights of the friction and of the form drag, we

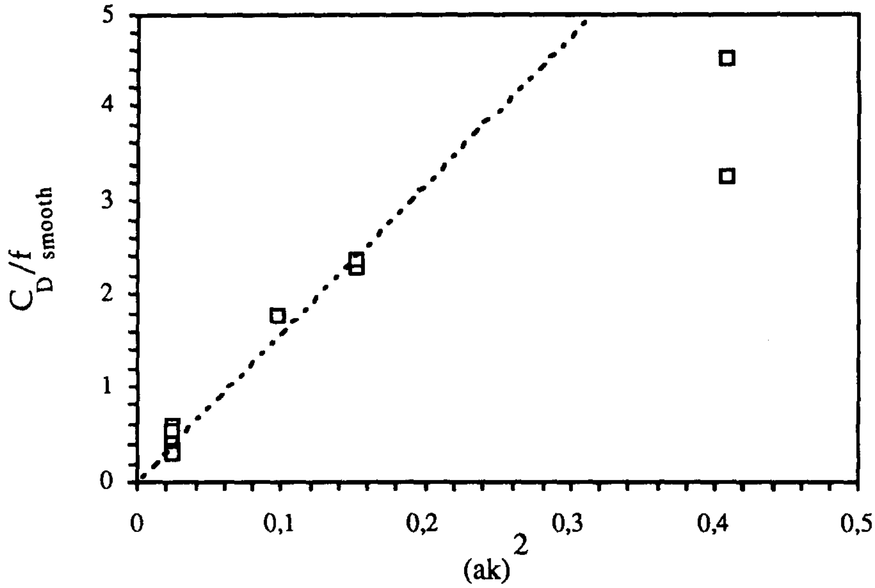

Figure 5. Experimental values of the ratio C_D/f_{smooth} vs $(ak)^2$ after Cook (1970), Kendall (1970), Zilker and Hanratty (1979).

will evaluate the two coefficients. Transposing the approach of Belcher and Hunt (1993) to our problem, we have plotted in figure 5 the ratio C_D/f_{smooth} vs $(ak)^2$. It is interesting to note that for small wave steepness, the trend is linear. In addition, the magnitude of the form drag coefficient is two or three times the magnitude of the friction factor f_{smooth}.

In the case of a wavy interface between gas and liquid, the problem is more complex. The wave is travelling at a celerity c. The characteristics of the dominant wave are unknown (amplitude a and wavenumber k). Ripples are generated at the surface of the wave and superimposed on the dominant wave, inducing a friction over a rough boundary (friction factor f_R, the subscript denoting the roughness as well as the ripples). It seems reasonable to look for a drag coefficient as a function of the following non-dimensional groups:

$$C_D = C_D(f_R, ak, c/V^*) \qquad [32a]$$

Considering the results over a stationary solid wavy wall, we have plotted in figure 6 the ratio $C_D/f_R/(ak)^2$ vs c/V^*, after the data of Shemdin and Hsu (1965). The values of the ratio c/V^* after Shemdin and Hsu (1965) are large since their experiments were carried out in a large flume with mechanically generated gravity waves. In stratified flow in pipes, the ratio is much smaller. Extrapolating the experimental trend given in figure 6 leads to a relatively constant value of the ratio $C_D/f_R/(ak)^2$. This value is in agreement with the trend given by figure 5 for stationary waves.

We propose the following quadratic dependence:

$$C_D = 14(ak)^2 f^i \qquad [32b]$$

Even if the friction factor can be related to a roughness scale given by a Charnock formula type, the problem remains open as long as the characteristics of the waves are not modelled.

In this paper, we have assumed a sinusoidal shape interface [17]. It leads to an analytical expression for the vertical profile of the liquid fraction [18]. In the Appendix, the phase distribution is shown to be determined by a momentum balance equation. In a general model of separated two-phase flow, [A6] enables us to calculate the vertical profile of the volume fraction of gas (or liquid) phase $\alpha^G(x_3) = 1 - \alpha^L(x_3)$. However, the calculation of $\alpha^G(x_3)$ is not sufficient to derive the shape of the gas–liquid interface. We have seen that a knowledge of both the shape and the stress distributions is required to calculate the interfacial transfer of momentum.

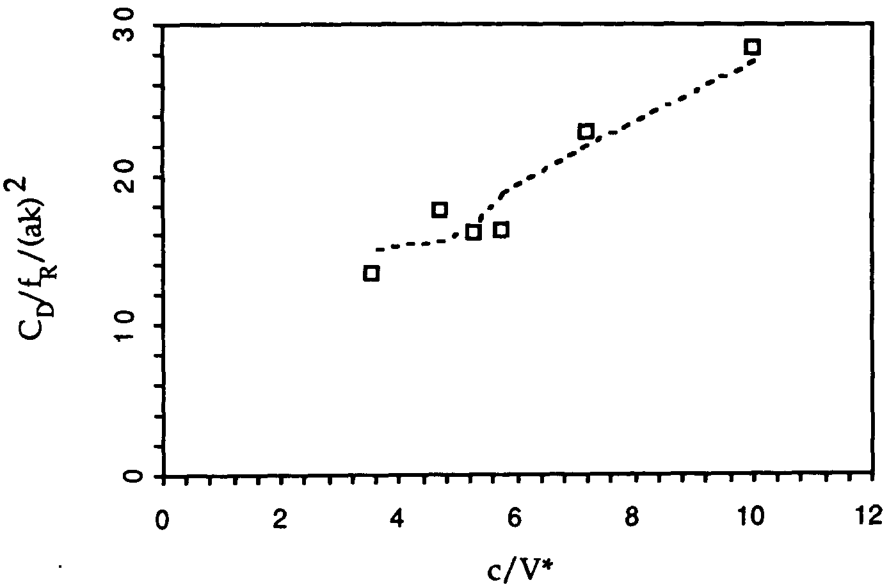

Figure 6. Experimental values of the ratio $C_D/f_R/(ak)^2$ vs c/V^* (□ data of Shemdin and Hsu (1965), --- Lopez theory (1995)).

5. CONCLUSION

Stratified flow is of importance in many industrial applications. In many previous works, the stratified flow has been considered as the superposition of two single-phase flows separated by a flat interface. It is pertinent when the liquid thickness is sufficiently large. However, when the liquid thickness decreases, the thickness of the wavy region cannot be neglected. Hence, a local two-fluid formulation is needed, the volume fraction of each phase decreasing continuously from 1 to 0 in the wavy region.

In this scope, the present study has two main objectives:

—first, the local two-fluid model of separated two-phase flow is formulated. The analysis focuses on momentum balance. Local momentum equations are written in a general form and discussed. It leads to a closure problem on interfacial momentum transfer. The cross-averaged momentum balance is also written in order to relate global and local approaches. The analysis of the set of equations of the local model shows that calculations of the velocity and pressure fields and phase distributions are required for the estimation of the interfacial term of momentum transfer. Basically, these terms contain the shape of the wave field and the distribution of the stresses at the interface;

—second, we have restricted our attention to a 2-D wave field. It is considered as the superimposition of a slowly varying dominant wave and small wavelength ripples. The linear perturbation caused by the sinusoidal wavy interface on the gas flow is analysed in terms of pressure and viscous stress variations. These stress perturbations are sinusoidal and phase shifted. We have focused on the pressure term which is shown from experiments to play a dominant role.

The drag coefficient was derived. It is related to the imaginary part of the pressure perturbation. The analysis of various experimental results shows that the form drag coefficient can be empirically correlated with a quadratic dependence with the wave slope.

Two issues have not been addressed in this paper:

—the mechanisms driving the generation of waves, their deformation and propagation have not been reviewed. Until now, there exists no model which is able to estimate this phenomenon. In a general model, the shape of the wavy interface given as a phenomenological relation must be compatible with the distribution of volume fraction of phase given by the solution of momentum balance [A6];

—the turbulent fields above and below the waves have also not been reviewed. Rapidly varying conditions above the waves (Belcher & Hunt 1993) as well as non-linear interactions between wave induced flow and turbulence field below the waves (Liné *et al.* 1996) require refined modelling of the turbulence.

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APPENDIX A

In order to simplify the discussion, we consider in this paragraph a fully-developed steady-state and quasi-parallel separated two-phase flow. It leads to a 1-D problem, the equations of the model being given by [17] and [18].

The two longitudinal momentum equation in [17] can be written for each phase. In fact, we can substitute into these two equations any linear combination of the longitudinal momentum equation of the mixture obtained by adding the two equations in [17]. It results in:

$$\frac{\partial(\alpha^G\overline{\overline{P^G}} + \alpha^L\overline{\overline{P^L}})}{\partial x_1} - \frac{\partial(\alpha^G(\overline{\overline{\tau^G_{31}}} + \rho^G\overline{\overline{\overline{R^G_{31}}}}) + \alpha^L(\overline{\overline{\tau^L_{31}}} + \rho^L\overline{\overline{\overline{R^L_{31}}}}))}{\partial x_3} - (\alpha^G\rho^G + \alpha^L\rho^L)g_1$$

$$+ (\langle P^i\rangle - \langle\tau^i_n\rangle)(\overline{n^G_1\delta^i} + \overline{n^L_1\delta^i}) - \langle\tau^i_t\rangle(\overline{t^G_1\delta^i} + \overline{t^L_1\delta^i}) = 0 \quad [A1]$$

[6] gives:

$$\overline{n^G_1\delta^i} = -\frac{\partial\alpha^G}{\partial x_1} \quad \text{and} \quad \overline{n^L_1\delta^i} = -\frac{\partial\alpha^L}{\partial x_1} = +\frac{\partial\alpha^G}{\partial x_1}, \quad \text{hence} \quad \overline{n^G_1\delta^i} + \overline{n^L_1\delta^i} = 0$$

In addition, one can write:

$$\overline{t^G_1\delta^i} = \overline{n^G_3\delta^i} = -\frac{\partial\alpha^G}{\partial x_3} \quad \text{and} \quad \overline{t^L_1\delta^i} = \overline{n^L_3\delta^i} = -\frac{\partial\alpha^L}{\partial x_3} = +\frac{\partial\alpha^G}{\partial x_3}, \quad \text{hence} \quad \overline{t^G_1\delta^i} + \overline{t^L_1\delta^i} = 0$$

Hence one obtains the classical momentum balance:

$$\frac{\partial(\alpha^G\overline{\overline{P^G}} + \alpha^L\overline{\overline{P^L}})}{\partial x_1} + \frac{\partial(\alpha^G(\overline{\overline{\tau^G_{31}}} + \rho^G\overline{\overline{\overline{R^G_{31}}}}) + \alpha^L(\overline{\overline{\tau^L_{31}}} + \rho^L\overline{\overline{\overline{R^L_{31}}}}))}{\partial x_3} - (\alpha^G\rho^G + \alpha^L\rho^L)g_1 = 0 \quad [A2]$$

The second equation can be obtained similarly by eliminating the pressure gradient in [17].

Similarly, the two vertical momentum equations in [18] can be written for each phase. Once more, we can substitute into these two equations any linear combination. The vertical momentum equation of the mixture is obtained by adding the two equations in [18]. It results in:

$$\frac{\partial(\alpha^G\overline{\overline{P^G}} + \alpha^L\overline{\overline{P^L}})}{\partial x^3} - \frac{\partial(\alpha^G\rho^G\overline{\overline{\overline{R^G_{33}}}} + \alpha^L\rho^L\overline{\overline{\overline{R^L_{33}}}})}{\partial x_3} - (\alpha^G\rho^G + \alpha^L\rho^L)g_3 = 0 \quad [A3]$$

which can be written after integration:

$$\overline{\overline{P}}(x_3) = \alpha^G\overline{\overline{P^G}} + \alpha^L\overline{\overline{P^L}} = -\alpha^G\rho^G\overline{\overline{\overline{R^G_{33}}}} + \alpha^L\rho^L\overline{\overline{\overline{R^L_{33}}}} - (\alpha^G\rho^G + \alpha^L\rho^L)g_3x_3 + cst \quad [A4]$$

Eliminating the pressure gradient gives:

$$\rho^G\frac{\partial\alpha^G\overline{\overline{\overline{R^G_{33}}}}}{\partial x_3} + \rho^G g_3 - \frac{1}{\alpha^G}(\overline{\overline{P^G}} - \langle P^i\rangle + \langle\tau^i_n\rangle)\frac{\partial\alpha^G}{\partial x_3} - \frac{1}{\alpha^G}\langle\tau^i_t\rangle\overline{t^G_3\delta^i} - \frac{1}{\alpha^G}[\overline{\tilde{P}n^G_3\delta^i} - \overline{\tilde{\tau}_n n^G_3\delta^i} - \overline{\tilde{\tau}_t t^G_3\delta^i}]$$

$$= -\frac{1}{\alpha^L}[\overline{\tilde{P}n^L_3\delta^i} - \overline{\tilde{\tau}_n n^L_3\delta^i} - \tilde{\tau}_t t^L_3\delta^i] + \rho^L\frac{\partial\alpha^L\overline{\overline{\overline{R^L_{33}}}}}{\partial x_3} + \rho^L g_3 - \frac{1}{\alpha^L}(\overline{\overline{P^L}} - \langle P^i\rangle + \langle\tau^i_n\rangle)\frac{\partial\alpha^L}{\partial x_3} - \frac{1}{\alpha^L}\langle\tau^i_t\rangle\overline{t^L_3\delta^i}$$

[A5]

In fully-developed flow conditions, one can write:

$$\overline{t_3^G \delta^i} = -\overline{n_1^G \delta^i} = -\frac{\partial \alpha^G}{\partial x_1} = 0 \quad \text{and} \quad \overline{t_3^L \delta^i} = \overline{n_1^L \delta^i} = -\frac{\partial \alpha^L}{\partial x_1} = 0$$

[A5] results in the following after manipulations:

$$\alpha^G \alpha^L \frac{\partial(\alpha^L \rho^L \overline{\overline{R_{33}^L}} - \alpha^G \rho^G \overline{\overline{R_{33}^G}})}{\partial x_3} + \alpha^G \alpha^L (\rho^L - \rho^G) g_3 + (\alpha^G \overline{\overline{P^L}} + \alpha^L \overline{\overline{P^G}} - \langle P^i \rangle + \langle \tau_n^i \rangle) \frac{\partial \alpha^G}{\partial x_3} + [\overline{\tilde{P} n_3^G \delta^i} - \overline{\tilde{\tau}_n n_3^G \delta^i} - \overline{\tilde{\tau}_t t_3^G \delta^i}] = 0 \quad \text{[A6]}$$